

The Topological Structure of Question Theory

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Abstract

A question is identified with a topology on a given set of irreducible assertions. It is shown that there are three types of a question. Type-I question generates sub-question, type-II question has a definite answer and type-III question is irrelevant. We suggest that the most intelligent machine asks type-II questions. We also claim that a truly intelligent machine cannot be desireless. This work may prove useful in machine learning and may open up new ways to understand mind.

1 Introduction

It has been pointed out by many authors that question is of fundamental importance [1, 2, 3, 4, 5, 6, 7, 8, 9]. In most cases we know the solution of a problem but we do not know the right question that answers it. It also happens sometime that a problem can be solved in many different ways. A truly intelligent machine is needed to ask relevant questions. It is however unimportant, at this stage, that some solutions are better than other.

A question is defined as the set of assertions that answers it. An older definition of the question is that a question is a request for information. The former definition was given by Richard T. Cox [1, 2], and studied further by others [3, 4, 5, 6, 7, 8, 9]. We follow R.T. Cox and define a question is a topology on a given set of irreducible assertions.

This paper is a self-contained development of the topological structure of question theory. Some definitions and theorems are given in section-2 which shall be used in the following sections. The paper is concluded in section-4.

2 Some Definitions and Theorems

The following definitions and theorems will be used in the sequel, see for instant [10].

Definition 2.1: *Let X be a non-empty set. A collection \mathcal{T} of subsets of*

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X is called a topology on X , if it meets the following requirements:

C1. ϕ and X belong to \mathcal{T} .

C2. The union of any number of sets in \mathcal{T} belong to \mathcal{T} .

C3. The intersection of any two sets (and hence of any finite number of set) in \mathcal{T} belongs to \mathcal{T} .

Definition 2.2: Let (X, \mathcal{T}) be a topological space. A subset U of X is said to be open iff it belongs to \mathcal{T} .

The complement $X - U$ is called closed set and a set which is both open and closed is called clopen.

Definition 2.3: Let (X, \mathcal{T}) be a topological space and let x be a point of X . A subset N of X is called a neighborhood (nbhd in short) of x iff there exists an open set u such that $x \in u \subseteq N \subseteq X$

Definition 2.4: Let x_0 be a point in a topological space (X, \mathcal{T}) , then the set of all nbhds is called nbhd system of the point x_0 and is denoted by $\mathcal{N}_{(x_0)}$.

Theorem 2.1: Let (X, \mathcal{T}) be a topological space and let $\mathcal{N}_{(x_0)}$ be the nbhd system of the point $x_0 \in X$, then

1. $\mathcal{N}_{(x_0)}$ is non-empty.
2. The intersection of any two members of $\mathcal{N}_{(x_0)}$ belongs to $\mathcal{N}_{(x_0)}$.
3. If U is in $\mathcal{N}_{(x_0)}$ and W is any set of X such that $U \subset W$, then W is in $\mathcal{N}_{(x_0)}$.

Definition 2.5: Let X be a topological space with topology \mathcal{T} . If A is subset of X , the collection

$$\mathcal{T}_A = \{A \cap U : U \in \mathcal{T}\}$$

is a topology on A , called the subspace topology, and A is called a subspace of X .

Definition 2.6: Let A and B be sets. The difference of A and B is the set of elements of A that are not in B , denoted by $A - B$. We write

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

3 Formalism

A question is formally defined as follow;

Definition 3.1 (Question): A question is identified with a topology \mathcal{T}_i on a given set X . The collection $\mathfrak{T} = \{\mathcal{T}_i : i \in I, \mathcal{T}_i \text{ is a topology on } X\}$ is the set of all possible questions that one can ask.

Definition-3.1 requires objective type questions. For illustration purpose consider the following objective-type question;

Q1: What is a particle?

- A a particle is something which has mass.
- B a particle is something which has spin.
- C a particle has both mass and spin.
- D none of the above.

Let mass = m , and spin = s , then the ground set for the above question is $X = \{m, s\}$. Question Q1 is described by the following topology

$$\mathcal{T}_1 = \{A, B, C, D\} = \{\{m\}, \{s\}, X, \phi\}$$

Let me ask the other possible questions in this space.

Q2: What is a massive particle?

A: having mass.

B: having mass and spin.

C: none of the above.

Q2 is described by the topology $\mathcal{T}_2 = \{\{m\}, X, \phi\}$. Similarly the question, 'What is a spinning particle?', is described by $\mathcal{T}_3 = \{\{s\}, X, \phi\}$. And the last question, 'What is a spinning-massive particle?', is described by $\mathcal{T}_4 = \{X, \phi\}$.

Every question consists of several issues. An issue is formally defined as follow;

Definition 3.2 (Issue): Any arbitrary point $x \in X$ be an irreducible assertion and the nbhd N of point x is said to be an issue related with point x and the nbhd system $\mathcal{N}_{(x)}$ is the set of all issues concerning x . The open nbhd is said to be a relevant issue for the current problem otherwise it is irrelevant.

Having defined a question and an issue, one has to resolve the issues of various points turn by turn by asking various questions. The following definition resolves an issue;

Definition 3.3 (Resolving an issue): Let (X, \mathcal{T}) be a topological space. $\mathcal{T} - \mathcal{N}_{(x_i)}$ resolves the issue of $x_i \in X$, where $\mathcal{N}_{(x_i)}$ is the neighborhood system of x_i .

Definition-3.3 tells us when an issue is solved than that issue no more exists in the space and hence we are not worried about it any more. One can see that the elements of $\mathcal{T} - \mathcal{N}_{(x_i)}$ do not contain x_i as a point. It is a kind of elimination procedure. The above definition leads to the following important theorem which may play essential role in machine learning. But let me first prove a lemma.

Lemma 3.1: Let $X \subseteq Y$, let \mathcal{T}_X be a topology on X , then there exists a topology \mathcal{T}_Y on Y , such that $\mathcal{T}_X \subseteq \mathcal{T}_Y$.

Said differently, every open set of X can be open in Y .

Proof Let \mathcal{T}_X and \mathcal{T}_Y be discrete topologies on X and Y respectively. Let $X \subseteq Y$. Since $X \cap Y = X$ ($X \subseteq Y$ by hypothesis) also given that \mathcal{T}_X and \mathcal{T}_Y are discrete, therefore X belongs to \mathcal{T}_Y and hence every member of \mathcal{T}_X is also a member of \mathcal{T}_Y . Hence $\mathcal{T}_X \subseteq \mathcal{T}_Y$.

Theorem 3.1: Let (X, \mathcal{T}_X) be a topological space,

- i) If $x_i \in X$ belongs to some but not all open sets of X then $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is a subspace topology of X .
- ii) If each non-empty open set of X contains x_i as a point, then $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is the singleton $\{\phi\}$.
- iii) If there exists no open set that contains x_i as a point, then $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is empty.

Proof i) Let $U = \{u_\alpha : \alpha \in I\}$ and $V = \{v_\alpha : \alpha \in I\}$ be subcollections of \mathcal{T}_X such that $x_i \in u_\alpha$ for every $u_\alpha \in U$ and $x_i \notin v_\alpha$ for every $v_\alpha \in V$. Let

$A = \bigcup_{v \in V} v$, to show (a) $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is a topology on A , (b) A is subspace of X .

First prove part-a

C1: Let $x_i \in u_\alpha \in U$ and $x_i \notin v_\alpha \in V$. It implies that u_α is a nbhd of x_i , therefore $u_\alpha \in \mathcal{N}_{(x_i)}$ and $x_i \notin v_\alpha$ implies that v_α is not a nbhd of x_i , therefore $v_\alpha \notin \mathcal{N}_{(x_i)}$. Hence $u_\alpha \notin \mathcal{T}_X - \mathcal{N}_{(x_i)}$ and $v_\alpha \in \mathcal{T}_X - \mathcal{N}_{(x_i)}$. Hence $A = \bigcup_{v \in V} v \in \mathcal{T}_X - \mathcal{N}_{(x_i)}$. Since $\mathcal{N}_{(x_i)}$ is non-empty (by theorem-2.1) and ϕ

belongs to \mathcal{T}_X (by theorem-2.1(C1)). Therefore $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ also contains ϕ .

C2: Given that $x_i \notin v_\alpha$, therefore $x_i \notin \bigcup_{\alpha} v_\alpha \in \mathcal{T}_X - \mathcal{N}_{(x_i)}$.

C3: Similarly $x_i \notin v_1 \cap v_2 \in \mathcal{T}_X - \mathcal{N}_{(x_i)}$.

Hence $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is a topology on A .

(b) It is shown in part-a that $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is a topology on A . Further $A \subseteq X$, therefore $\mathcal{T}_X - \mathcal{N}_{(x_i)} = \{A \cap U : U \in \mathcal{T}_X\}$ is a subspace topology of X (by the definition of subspace topology).

ii) Let $x_i \in u$ for every non-empty set $u \in \mathcal{T}_X$, then by theorem-2.1 each u is contained in $\mathcal{N}_{(x_i)}$. Since $\mathcal{N}_{(x_i)}$ is non-empty hence ϕ is the only member of \mathcal{T}_X which is not in $\mathcal{N}_{(x_i)}$. Therefore $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is the singleton $\{\phi\}$.

iii) Let $x_i \notin u$ for every $u \in \mathcal{T}_X$ and assume $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is non-empty. When $x_i \notin u$, then there exists no nbhd N of x_i such that $N \in \mathcal{N}_{(x_i)}$. Therefore $\mathcal{N}_{(x_i)}$ is empty which is a contradiction (because $\mathcal{N}_{(x_i)}$ is non-empty by theorem-2.1). Hence $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is empty.

Converse of the theorem is not unique by lemma-3.1 in that $\mathcal{T}_X - \mathcal{N}_{(x_i)} \subseteq \mathcal{T}_X \subseteq \mathcal{T}_Y$.

Theorem-3.1 has three parts. It admits three types of questions i.e., type-I, type-II and type-III given by cases (i), (ii) and (iii) respectively. Type-I question generates a sub-question, type-II question has a definite answer and type-III question is irrelevant. Here if $\mathcal{T} - \mathcal{N}_{(x_i)}$ ends up with sub-question (a sub-question is identified with a subspace), then one may ask further questions we call them sub-questions. When $\mathcal{T} - \mathcal{N}_{(x_i)}$ is equal to $\{\phi\}$, it means that the question has a definite answer and thus no further questions are needed to ask and when $\mathcal{T} - \mathcal{N}_{(x_i)}$ is empty then the question was irrelevant. To better understand various types of a question, let me increase the size of the space in the previous example, by including charge of the particle. Let $X = \{m, s, e\}$, let $\mathcal{T}_X = \{\{m\}, \{m, s\}, \{m, e\}, X, \phi\}$ which corresponds to the question; *What is a massive particle?* I am interested to resolve the issue of charge, then

$$\begin{aligned} \mathcal{T}_X - \mathcal{N}_{(e)} &= \{\{m\}, \{m, s\}, \{m, e\}, X, \phi\} - \{\{m, e\}, X\} \\ &= \{\{m\}, \{m, s\}, \phi\} \end{aligned}$$

Since $\mathcal{T}_X - \mathcal{N}_{(e)}$ is a topology on $\{m, s\}$, therefore it corresponds to the sub-question; *'What is a neutral-massive particle?'* One can see that $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is an elimination procedure, i.e. $\mathcal{T}_X - \mathcal{N}_{(x_i)}$ is a collection of subset of X which does not contain x_i as a point. In the above example, when the issue of charge is solved then the charge no more exists in the subspace. If we remove the nbhd

system of m , then

$$\mathcal{T}_X - \mathcal{N}_{(m)} = \{\phi\}$$

Since $\mathcal{T}_X - \mathcal{N}_{(m)}$ describes type-II question. Hence a massive particle has a definite mass and so no further questions are needed to ask. Now consider type-III question, suppose it is given that the motion of a particle is described by classical mechanics, then asking about the quantum mechanical nature of the particle is irrelevant.

The converse of the theorem is not unique means that while starting from a given solution one can generate several higher order questions.

3.1 Most and least efficient questions

The operation $\mathcal{T} - \mathcal{N}_{(x_i)}$ acts most efficiently and terminates the processes in just one single step, when theorem-3.1(ii) is true. In the collection $\mathfrak{T} = \{\mathcal{T}_j : j \in I\}$, there exist \mathcal{T}_k such that $\mathcal{T}_k - \mathcal{N}_{(x_i)} = \{\phi\}$. If X is discrete, then $\mathcal{T} - \mathcal{N}_{(x_i)}$ operates least efficiently. When X has more than one points then discrete topologies satisfy the first case of theorem-3.1. When X is a singleton then it satisfies both first and second cases of theorem-3.1. One can observe that when X is discrete, then the subspace topology $\mathcal{T} - \mathcal{N}_{(x_i)}$ is a topology on $X - \{x_i\}$. Thus $\mathcal{T} - \mathcal{N}_{(x_i)}$ eliminates just one irreducible assertion from the space. There is a topological property, called the hereditary property, whenever a topological space X has property P , then every subspace of X also has property P . It implies that $\mathcal{T} - \mathcal{N}_{(x_i)}$ is a discrete topology on $X - \{x_i\}$. Let \mathcal{T} be a discrete topology on X , define $\mathcal{T}_2 = \mathcal{T} - \mathcal{N}_{(x_i)}$, then $\mathcal{T}_2 - \mathcal{N}_{(x_i)}$ is a discrete topology on $X - \{x_i\}$, such a process is very slow. Every time just one irreducible assertion is eliminated from the space.

3.2 Negation question

A question can also be asked negatively. Consider two systems A and B which are thermally connected. Let the energy of A is greater than the energy of B (i.e., $E_A > E_B$), then the energy is flowing from A to B . In this case system A asks from system B , "Do you want to gain energy?" In the response system B asks from system A , "Don't you want to gain energy?" If we define the former question is to be a topology \mathcal{T}_X^U , then the later question is defined as follow;

Definition 3.4 (negation question): Let \mathcal{T}_X^U be a topology on X . The corresponding negation question is defined as $\mathcal{T}_X^F = \{X - U : U \in \mathcal{T}_X^U\}$.

One can show that \mathcal{T}_X^F is also a topology on X and that $\mathcal{T}_X^U \neq \mathcal{T}_X^F$ in general. The intersection $\mathcal{T}_X^U \cap \mathcal{T}_X^F$ is the collection of clopen sets. When every member of \mathcal{T}_X^U is clopen, then $\mathcal{T}_X^U = \mathcal{T}_X^F$. In this case question and its negation question are identical. It is believed that question and negation question are always identical [6]. It is argued that both questions are identical in the sense that they ask the same thing. It is demonstrated by asking, 'Is it raining?' and 'Is it not raining?' Both questions are answered by the statements 'It is raining!' and

'It is not raining!'; and thus they are equivalent. Here an important question is missing, that is, 'Who asks the question?'. Let me replace the raining question by the following simple question and demonstrate that they are not equivalent. Consider Bob and Alice. Let Bob has extra money and Alice needs some money. Bob wants to get rid of extra money and asks Alice, "Do you need money?" Alice needs the money she would respond, "Don't you need the money?" The two persons ask questions according to their desires and thus generate different questions. Here money is flowing from one person to the other. We conclude from this that for a machine (Machine is defined below) to be intelligent it must have desires. A clear example of an intelligent machine who has got desires is human. Let me put it in this way humans are intelligent because they have desires. A truly artificially intelligent machine can think independently when it is able to create desires which forces it to act accordingly. We also claim that desires are quantifiable.

The equivalence of question and negation question make sense in probability theory. A probability space is a triple (Ω, \mathcal{F}, P) , where \mathcal{F} is a collection of subsets of Ω relevant to a particular experiment and P is a probability measure. The elements of \mathcal{F} are called events. A non-empty collection of subsets \mathcal{F} is called a σ -field on Ω if [11]

1. $\phi \in \mathcal{F}$
2. $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where A^c is compliment of A .
3. $A_i \in \mathcal{F}$, $i \in I$, I = a countable set, then $\bigcup_{i \in I} A_i \in \mathcal{F}$.

The last condition implies $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$ (by De Morgan's formula).

Therefore $\bigcap_{i \in I} A_i^c \in \mathcal{F}$. In our case it corresponds to $\mathcal{T}_X^U = \mathcal{T}_X^F$.

3.3 Machine

The negation question is very important for construction of a machine. A machine is defined as an agent who asks question \mathcal{T}_X^U . For every machine there exists an anti-machine who asks the negation question \mathcal{T}_X^F . A machine and an anti-machine make a universe. Here 'machine' is used in a very broad sense. It can be an intelligent machine which ask a question that has definite answer (type-II questions) or it can be a system of particles and anti-machine is the rest of the universe or the environment. We suspect that noise is also a kind of anti-machine. The two machines communicates through the clopen sets. Said differently the clopen sets are the information the two machines share to operate and the non-clopen set are the information they do not share. The two machine are in perfect agreement when they share all informations. In this case $\mathcal{T}_X^U = \mathcal{T}_X^F$.

The collection $\mathfrak{T} = \{\mathcal{T}_i : i \in I\}$ is the set of all possible questions that a machine can ask. Therefore a machine consists of several components (subsystems). Each component is called an atomic machine. Each atomic machine asks a question $\mathcal{T}_i^U \in \mathfrak{T}$, and for every atomic machine there exists an anti-atomic

machine who asks the negation question $\mathcal{T}_i^F \in \mathfrak{T}$. Some atomic machine are their own anti-atomic machine for which $\mathcal{T}_j^U = \mathcal{T}_j^F \in \mathfrak{T}$.

4 Conclusion

It is possible to construct a machine that asks intelligent questions and work most efficiently. We suggest a machine work most efficiently if it asks type-II question. This paper is a qualitative development of question theory. Our formalism is general, it can be applied to any problem where question exists.

Acknowledgements

I am indebted to A. Caticha, A. Inomata and S. Ali for many insightful remarks and valuable suggestions. I also benefited from K. Knuth's thoughtful lectures on Question Algebra.

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